Variational Principle of Melting Solid with Three Phase.

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The heat conduction of the melting solid with three phase; that is the solid state, the melted state and the vaporized state, is investigated by variational method. The variational equation of heat conduction with three phase is formulated. This equation has two parameters, i. e. the position of vaporizing line $s_1(t)$ and melting line $s_2(t)$, therefore the energy equation is used as a subsidary condition. A quadratic formula is used as a test function. The variational equation with subsidary condition together with the test function provide a series solution of $s_1(t)$ and $s_2(t)$.

§1. Introduction

In the previous works, we concerned with the melting elastic solid.[1),2)] Investigetion was done by using the variational method of thermoelasticity.[3)−5)]

In the present paper, we shall investigate the melting sloid with three phase, i. e. the solid state, the melted state and the vaporized state. In this case the temperture field has two parameters, that is the position of vaporizing line $s_1(t)$ and melting line $s_2(t)$. Therefore, if we evaluate this problem by variational method, we must formulate a variational equation with subsidary condition. We choose the energy equation as[6),7)] the subsidary condition.

The variational equation and the energy equation do not require the evaluation of temperature field. So, we can use a test function including $s_1(t)$ and $s_2(t)$ as parameters. The quadratic formula is chosen as a test function. Substitution of the test function into the variational equation and the energy equation provides a system of differential equation of $s_1(t)$ and $s_2(t)$, which we may find the series solution with the aid of Adams-Bashforth's method.

§2. Variational Principle

The problem of melting and vaporizing slab may be considered. Consider a slab occupying the region $(0, L)$, exposed to a prescribcd heat input $Q(t)$ at $x=0$, and the other end $x=L$ is perfectly insulated. Let $T+\theta$ denote the temperature of unit element, θ being the temperature change. It will be assumed that the slab is firstly melted and secondly vaporized. We assume that the position of vaporizing line is $x=s_1(t)$ and melting line is $x=s_2(t)$.

To formulate a variational equation, we choose the variational invariant

$$V=\int_{s_1}^{s_2}\frac{\lambda_M}{2}\left(\frac{\partial\theta_M}{\partial x}\right)^2 dx+\int_{s_2}^{L}\cdot\frac{\lambda_S}{2}\left(\frac{\partial\theta_S}{\partial x}\right)^2 dx, \tag{2.1}$$

where λ is heat conductivity, suffixes M and S denote the melted state and the solid state, respectively.

We take the variations as the changes due to the virtual displacement of $s_1(t)$ and $s_2(t)$. The variation of V is calculated as

$$\begin{aligned}\delta V=&\frac{\lambda_m}{2}\left(\frac{\partial\theta_M}{\partial x}\right)_m^2\delta s_2-\frac{\lambda_v}{2}\left(\frac{\partial\theta_M}{\partial x}\right)_v\delta s_1\\&+\int_{s_1}^{s_2}\lambda_M\frac{\partial\theta_M}{\partial x}\frac{\partial}{\partial x}(\delta\theta_M)\,dx-\frac{\lambda_m}{2}\left(\frac{\partial\theta_S}{\partial x}\right)_m\delta s_2\\&+\int_{s_2}^{L}\lambda_S\frac{\partial\theta_S}{\partial x}\frac{\partial}{\partial x}(\delta\theta_S)\,dx,\end{aligned} \tag{2.2}$$

where suffixes m and v denote the melting state and vaporizing state respectively. The boundary conditions at $x=s_1(t)$ and $x=s_2(t)$ are

$$-\lambda_v\left(\frac{\partial\theta_M}{\partial x}\right)_v=Q(t)-(\rho l)_1\ \dot{s}_1, \tag{2.3}$$

$$-\lambda_m\left(\frac{\partial\theta_S}{\partial x}\right)_m=-\lambda_m\left(\frac{\partial\theta_M}{\partial x}\right)_m-(\rho l)_2\ \dot{s}_2, \tag{2.4}$$

where $(\rho l)_1$, and $(\rho l)_2$ denote the latent heat of vaporization and melting per unit volume respectively.

The boundary condition at $x=L$ is

$$\left(\frac{\partial\theta_S}{\partial x}\right)_{x=L}=0. \tag{2.5}$$

The heat conduction equation is

$$c\dot{\theta}=\frac{\partial}{\partial x}\left(\lambda\frac{\partial\theta}{\partial x}\right). \tag{2.6}$$

where c is the heat capacity per unit volume.

Using the fact

$$(\delta\theta_M)_v=(\delta\theta_M)_m=0,\qquad(\delta\theta_S)_m=0$$

and eqs. (2.5) and (2.6), we find

$$\int_{s_1}^{s_2}\lambda_M\frac{\partial\theta_M}{\partial x}\frac{\partial}{\partial x}(\delta\theta_M)\,dx=-\int_{s_1}^{s_2}c_M\dot{\theta}_M\,\delta\theta_M\,dx \tag{2.7}$$

$$\int_{s_2}^{L}\lambda_S\frac{\partial\theta_S}{\partial x}\frac{\partial}{\partial x}(\delta\theta_S)\,dx=-\int_{s_2}^{L}c_S\dot{\theta}_S\,\delta\theta_S\,dx. \tag{2.8}$$

Substituting eqs. (2.3), (2.4), (2.7) and (2.8) into eq. (2.2), we find

$$\delta V=-\frac{1}{2}(\rho l)_2\ s_2\left[\left(\frac{\partial\theta_M}{\partial x}\right)_m+\left(\frac{\partial\theta_S}{\partial x}\right)_m\right]\delta s_2+\frac{1}{2}\left[Q(t)-(\rho l)_1\ \dot{s}_1\right]\left(\frac{\partial\theta_M}{\partial x}\right)_v\delta s_1$$

$$-\int_{s_1}^{s_2} c_M\,\dot{\theta}_M\,\delta\theta_M\,dx-\int_{s_2}^{L} c_S\,\dot{\theta}_S\ \delta\theta_S\ dx\ . \tag{2.9}$$

We choose the second variational invariant D such that

$$\delta D=\int_{s_1}^{s_2} c_M\dot{\theta}_M\ \delta\theta_M\ dx+\int_{s_2}^{L} c_S\dot{\theta}_S\ \delta\theta_S\ dx\ . \tag{2.10}$$

The equations (2.9) and (2.10) provide the variational principle

$$\delta V+\delta D=-\frac{1}{2}(\rho l)_2\ \dot{s}_2\left[\left(\frac{\partial\theta_M}{\partial x}\right)_m+\left(\frac{\partial\theta_S}{\partial x}\right)_m\right]\delta s_2$$

$$+\left[Q(t)-(\rho l)_1\ \dot{s}_1\right]\left(\frac{\partial\theta_M}{\partial x}\right)_v\ \delta s_1\ . \tag{2.11}$$

This equation does not require the evaluation of temperature field, therefore we use a test function satisfying the boundary condition.

The test function is chosen as a quadratic formula

$$\begin{cases}\theta_M=\theta_m+(\theta_v-\theta_m)\dfrac{(s_2-x)^2}{(s_2-s_1)^2},\\[2ex] \theta_S=\theta_m\dfrac{(L-x)^2}{(L-s_2)^2}\end{cases} \tag{2.12}$$

Substituting eq. (2.12) into eq. (2.1), we find

$$V=\frac{2\lambda_M}{3}\ \frac{(\theta_v-\theta_m)^2}{s_2-s_1}+\frac{2\lambda_S\,\theta^2{}_m}{3}\ \frac{1}{L-s_2}$$

The variation of V is derived, such that

$$\delta V=-\frac{2\lambda_M}{3}\ \frac{(\theta_v-\theta_m)^2}{(s_2-s_1)^2}\ \delta(s_2-s_1)+\frac{2\lambda_S\,\theta^2{}_m}{3}\ \frac{1}{(L-s_2)^2}\ \delta s_2\ . \tag{2.13}$$

The eq. (2.12) provide

$$\begin{cases}\dot{\theta}_M=2(\theta_v-\theta_m)\left[\dfrac{(s_2-x)\,\dot{s}_2}{(s_2-s_1)^2}-\dfrac{(s_2-x)^2(\dot{s}_2-\dot{s}_1)}{(s_2-s_1)^3}\right],\\[2ex] \delta\theta_M=2(\theta_v-\theta_m)\left[\dfrac{s_2-x}{(s_2-s_1)^2}\ \delta s_2-\dfrac{(s_2-x)^2}{(s_2-s_1)^3}\ \delta(s_2-s_1)\right],\end{cases}$$

$$\begin{cases}\dot{\theta}_S=2\theta_m\dfrac{\dot{s}_2}{(L-s_2)^3}\ (L-x)^2,\\[2ex] \delta\theta_S=2\theta_m\dfrac{(L-x)^2}{(L-s_2)^3}\ \delta s_2\ .\end{cases}$$

Substituting above equations jnto eq. (2.10), we find δD, such that

$$\delta D = c_M \frac{(\theta_v - \theta_m)^2}{s_2 - s_1} \left[\frac{1}{3}(\dot{s}_2 + 3\dot{s}_1)\ \delta s_2 + \frac{1}{5}(\dot{s}_2 - 4\dot{s}_1)\delta(s_2 - s_1) \right]$$

$$+ \frac{4 c_v \theta^2{}_m}{5} \frac{\dot{s}_2}{L - s_2} \delta s_2 \ , \qquad (2.14)$$

Inserting eq. (2.12), (2.13) and (2.14) into the variational equation (2.11), we see

$$\left[\frac{\theta_m (\rho l)_2}{L - s_2} \dot{s}_2 - (\theta_v - \theta_m) \frac{Q(t) - (\rho l)_1 \dot{s}_1}{s_2 - s_1} \right] \delta s_2$$

$$+ (\theta_v - \theta_m) \frac{Q(t) - (\rho l)_1 \dot{s}_1}{s_2 - s_1} \delta(s_2 - s_1)$$

$$= \left[-\frac{2\lambda_M}{3} \frac{(\theta_v - \theta_m)^2}{(s_2 - s_1)^2} + c_M \frac{(\theta_v - \theta_m)^2}{s_2 - s_1} \frac{1}{5}(\dot{s}_2 - 4\dot{s}_1) \right] \delta(s_2 - s_1)$$

$$+ \left[\frac{2\lambda_S \theta^2{}_m}{3} \frac{1}{(L - s_2)^2} + c_M \frac{(\theta_v - \theta_m)^2}{s_2 - s_1} \frac{1}{3}(\dot{s}_2 + 3\dot{s}_1) \right.$$

$$\left. + \frac{4 c_S \theta^2{}_m}{5} \frac{\dot{s}_2}{L - s_2} \right] \delta s_2 \qquad (2.15)$$

This equation has parameters $s_2 - s_1$ and s_2, therefore we must find the subsidary condition which connect the relation between $\delta(s_2 - s_1)$ and δs_2 .

§ 3. Energy Equation

As in the previous paper, we take the origin of the time as the time when the vaporization begins, that is $s_1(0) = 0$.

The energy equation[6),7)] is formulated such as

$$\int_0^t Q(t)\ dt + \Delta = (\rho l)_2 s_2 + (\rho l)_1 s_1 + c_S \theta_m s_2$$

$$+ c_M(\theta_v - \theta_m) s_1 + \int_{s_1}^{s_2} c_M(\theta_M - \theta_m) dx + \int_{s_2}^{L} c_S \theta_S\ dx\ . \qquad (3.1)$$

where Δ denotes the heat transported to right across the unit cross-sectional area at $x=0$, before the vaporization begins. This equation also does not require the evaluation of temperature field, so we can use the test function (2.12). Inserting eq. (2.12) into eq. (3.1), we find

$$\int_0^t Q(t) dt + \Delta = B s_2 + A(s_2 - s_1) + \frac{1}{3} c_S \theta_m L,$$

with

$$A = \frac{1}{3} c_M(\theta_v - \theta_m) - (\rho l)_1 - c_v(\theta_v - \theta_m),$$

$$B = (\rho l)_2 + c_S \theta_m + (\rho l)_1 + c_M(\theta_v - \theta_m) - \frac{1}{3} c_S \theta_m \qquad (3.2)$$

The equation (3.2) provides

$$\delta(s_2 - s_1) = -\frac{\mathrm{B}}{\mathrm{A}}\,\delta s_2 . \tag{3.3}$$

Substitution of eq. (3.3) into eq. (2.15) provides

$$\frac{\theta_m(\rho l)_2}{L-s_2}s_2 - (\theta_v - \theta_m)\frac{Q(t)-(\rho l)_1\dot{s}_1}{s_2-s_1} - \frac{\mathrm{B}}{\mathrm{A}}(\theta_v - \theta_m)\frac{Q(t)-(\rho l)_1\dot{s}_1}{s_2-s_1}$$
$$= -\frac{\mathrm{B}}{A}\left[-\frac{2\lambda_M}{3}\frac{(\theta_v-\theta_m)^2}{(s_2-s_1)^2} + \frac{c_M}{5}\frac{(\theta_v-\theta_m)^2}{s_2-s_1}(\dot{s}_2-4\dot{s}_1)\right]$$
$$+\left[\frac{2\lambda_S\theta_m^2}{3}\frac{1}{(L-s_2)^2} + \frac{c_M}{3}\frac{(\theta_v-\theta_m)^2}{s_2-s_1}(\dot{s}_2+3\dot{s}_1)\right.$$
$$\left.+\frac{4c_S\theta_m^2}{5}\frac{s_2}{L-s_2}\right]. \tag{3.4}$$

The equations (3.2) and (3.4) give us the system of equations for s_1 and s_2.

§4. Series Solution

From eqs. (3.4) and (3.2), we find the series solution of s_1 and s_2 with aid of Adams-Bashforth's method:

$$\begin{cases} s_1(t) = \dot{s}_1(0)t + \dfrac{1}{2!}\ddot{s}_1(0)t^2 + \cdots\cdots, \\ s_2(t) = s_2(0) + \dot{s}_2(0)t + \dfrac{1}{2!}\ddot{s}_2(0)\,t^2 + \cdots\cdots, \end{cases}$$

with

$$s_2(0) = \frac{1}{A+B}\left(\Delta - \frac{1}{3}c_S\theta_m L\right),$$

$$\begin{cases} \dot{s}_2(0) = \dfrac{AE + D\cdot Q(0)}{CA + D\cdot(A+B)}, \\ \dot{s}_1(0) = -\dfrac{CQ(0) - E(A+B)}{CA + D\cdot(A+B)}, \end{cases}$$

$$\begin{cases} \ddot{s}_2(0) = \dfrac{G\dot{Q}(0) + AH}{G(A+B)+AF} \\ \ddot{s}_1(0) = \dfrac{H(A+B) - \dot{Q}(0)F}{G(A+B)+AF}, \end{cases}$$

$$C = \frac{\theta_m(\rho l)_2}{L-s_2(0)} + \frac{B}{A}\frac{c_M(\theta_v-\theta_m)^2}{5s_2(0)} - \frac{c_M}{3}\frac{(\theta_v-\theta_m)^2}{s_2(0)} - \frac{4c_S\theta_m^2}{5L},$$

$$D = \frac{(\rho l)_1(\theta_v-\theta_m)}{s_2(0)} + \frac{B}{A}\frac{(\theta_v-\theta_m)(\rho l)_1}{s_2(0)} - \frac{4c_M(\theta_v-\theta_m)^2}{5s_2(0)}\frac{B}{A} - c_M\frac{(\theta_v-\theta_m)^2}{s_2(0)}$$

$$E=(\theta_v-\theta_m)\frac{Q(0)}{s_2(0)}+\frac{B}{A}(\theta_v-\theta_m)\frac{Q(0)}{s_2(0)}+\frac{B}{A}\frac{2\lambda_M}{3}\frac{(\theta_v-\theta_m)^2}{s_2(0)^2}$$

$$+\frac{2\lambda_S\theta_m^2}{3}\frac{1}{(L-s_2(0))^2},$$

$$F=\frac{\theta_m(\rho l)_2}{L-s_2(0)}+\frac{B}{A}\frac{c_M}{5}\frac{(\theta_v-\theta_m)^2}{s_2(0)}-\frac{c_M}{3}(\theta_v-\theta_m)^2\frac{1}{s_2(0)}-\frac{4c_S\theta_m^2}{5}\frac{1}{L-s_2(0)}$$

$$G=\frac{(\theta_v-\theta_m)(\rho l)_1}{s_2(0)}\left(1+\frac{B}{A}\right)+\frac{B}{A}\frac{4c_M}{5}\frac{(\theta_v-\theta_m)^2}{s_2(0)}-\frac{c_M(\theta_v-\theta_m)^2}{s_2(0)}$$

$$H=-\frac{\theta_m(\rho l)_2}{(L-s_2(0))^2}\dot{s}_2(0)^2+(\theta_v-\theta_m)\left(1+\frac{B}{A}\right)\left[\frac{\dot{Q}(0)}{s_2(0)}\right.$$

$$\left.-\frac{Q(0)-(\rho l)_1\dot{s}_1(0)}{s_2(0)^2}(\dot{s}_2(0)-\dot{s}_1(0)\right]-\frac{B}{A}\left[\frac{4\lambda_M}{3s_2(0)^3}(\dot{s}_2(0)-\dot{s}_1(0))\right.$$

$$-\frac{c_M(\theta_v-\theta_m)^2}{5s_2(0))^2}(\dot{s}_2(0)-4\dot{s}_1(0))(\dot{s}_2(0)-\dot{s}_1(0)+\frac{4\lambda_S\theta_m^2}{3}\frac{\dot{s}_2(0)}{(L-s_2(0))^3}$$

$$-\frac{c_M(\theta_v-\theta_m)^2}{3s_2(0)^2}(\dot{s}_2(0)+3\dot{s}_1(0))(\dot{s}_2(0)-\dot{s}_1(0))+\frac{4c_S\theta_m^2}{5}\frac{\dot{s}_2(0)^2}{(L-s_2(0))^2}$$

§5. Conclusion

The results may be summerized as follows:

1) To formulate the variational equation, we have chosen the variational invariants V and D as eqs. (2.1) and (2.10).

2) By taking the variations as the changes of the quantities due to the virtual displacement of the position of the vaporizing line and the melting line, we have found the variational equation (2.11).

3) The energy equation (3.1) has been chosen as the subsidary condition.

4) The quadratic formula (2.12) has been used as the test function.

5) The variational equation (2.11) with the subsidary condition (3.1) together with the test function (2.12) provides the series solution.

References

1) Y. Furuya: J. Phys. Soc. Japan **43** (1977) 1068.
2) Y. Furuya: J. Phys. Soc. Japan **45** (1978) 1015.
3) M. Biot: J. appl. Phys. **27** (1956) 240.
4) M. Biot: J. Aero. Sci. **24** (1957) 857.
5) M. Biot: Indiana Univ. Math. J. **23** (1973) 309.
6) B. Boley: Quart. appl. Math. **21** (1963) 1.
7) B. Boley: Quart. J. Mech. appl. Math. **17** (1964) 253.

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